Bipartite Distance-Regular Graphs of Valency Three

Tatsuro Ito
Department of Mathematics
Ohio State University
Columbus, Ohio 43210

and

Department of Mathematics
Tsukuba University
Sakuramura, Ibaraki, Japan

Submitted by N. Biggs

ABSTRACT

Bipartite distance-regular graphs of valency three are classified. There are eight such graphs, all of which have diameter less than 9, and seven of them are distance-transitive.

1. INTRODUCTION

It has been conjectured that the diameter of any distance-regular graph is bounded by a function depending only on the valency (if the valency is not 2). In this paper we prove a very special case of this conjecture. We shall prove that every *bipartite* distance-regular graph of valency 3 has diameter less than 9. We also give a complete list of these graphs. Our main theorem is:

THEOREM. *Let* Γ *be a bipartite distance-regular graph of valency* 3. *Then* Γ *is one of the eight graphs listed in Table* 1. *Each of these graphs is unique up to isomorphism.*

Graphs (v) and (vii) were described in [2] and [9]. Graph (iv) is a 3-fold covering of the complete bipartite graph $K_{3,3}$, graph (vi) is a double covering of O_3 (Petersen's graph), and graph (viii) is a 3-fold covering of graph (v).

The conjecture was suggested by the corresponding results on distance-transitive graphs. If a graph is distance-transitive, it has a large group of automorphisms, and many powerful theorems on distance-transitive graphs

52 Vanderbilt Ave., New York, NY 10017 0024-3795/82/050195+19$02.75

TABLE 1

	Name	No. of vertices	Girth	Diameter
(i)	$K_{3,3}$	6	4	2
(ii)	Cube	8	4	3
(iii)	PG(2, 2)	14	6	3
(iv)	$3 \cdot K_{3,3}$	18	6	4
(v)	generalized 4-gon	30	8	4
(vi)	$2 \cdot O_3$	20	6	5
(vii)	generalized 6-gon	126	12	6
(viii)	$3 \cdot (v)$	90	10	8

were proved by using the theory of permutation groups [4, 10]. It is a remarkable fact that although distance-regularity is a much weaker condition than distance transitivity, only one graph in the above list—that is, (vii)—is not distance-transitive.

2. PRELIMINARIES

We begin with the definition of distance-regular graphs and pick up some fundamental properties of such graphs. Readers are referred to [3].

In this paper the term "graph" means a finite simple undirected graph, and the number of vertices of Γ is denoted by n. Take a vertex u, and let $\Lambda_i(u)$ denote the set of vertices which have distance i from u. Take a vertex $v \in \Lambda_i(u)$, and let

$$a_i = |\Lambda_i(u) \cap \Lambda_1(v)|,$$

$$b_i = |\Lambda_{i+1}(u) \cap \Lambda_1(v)|,$$

$$c_i = |\Lambda_{i-1}(u) \cap \Lambda_1(v)|.$$

In general, the numbers a_i, b_i, c_i depend on the choice of u and v as well as i. A graph Γ is said to be *distance-regular* if it is connected and a_i, b_i, c_i are independent of the choice of u and v.

In what follows, we always assume that Γ is distance-regular. The diameter of Γ is denoted by d, and the adjacency matrix of Γ is denoted by A. The

a_i, b_i, c_i are called intersection numbers and satisfy the following properties:

PROPOSITION 2.1 [3, Proposition 20.4].

(i) *Each of the b_i's and c_i's is nonzero, and $a_0 = 0$, $b_0 = |\Lambda_1(u)|$, $c_1 = 1$. $|\Lambda_1(u)|$ is denoted by k and called the valency of Γ.*

(ii) *$a_i + b_i + c_i = k$ for $i = 1, 2, \ldots, d-1$, and*

$$a_d + c_d = k.$$

(iii) *$k \geqslant b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{d-1}$, and*

$$1 \leqslant c_1 \leqslant c_2 \leqslant \cdots \leqslant c_d \leqslant k.$$

Γ is called *bipartite* if Γ has a partition of the vertex set into two subsets each of which contains no pair of adjacent vertices. In the case where Γ is distance-regular, Γ is bipartite if and only if all the a_i's are zero.

We call the following tridiagonal matrix B of size $d+1$ the *intersection matrix* of Γ:

$$B = \begin{bmatrix} 0 & 1 & & & & \bigcirc \\ k & a_1 & c_2 & & & \\ & b_1 & a_2 & \ddots & & \\ & & b_2 & \ddots & c_{d-1} & \\ & & & \ddots & a_{d-1} & c_d \\ \bigcirc & & & & b_{d-1} & a_d \end{bmatrix}.$$

PROPOSITION 2.2 [3, Proposition 21.2; 6, (6.6)].

(i) *The algebra spanned by A over $\mathbb{C}$ is isomorphic to that spanned by B. In particular, A and B have the same minimal polynomial.*

(ii) *The minimal polynomial of B is $(x-k)F_d(x)$, where $F_d(x)$ is a polynomial of degree d determined by the three term recursion*

$$F_i(x) = (x - k + c_i + b_{i-1})F_{i-1}(x) - b_{i-1}c_{i-1}F_{i-2}(x)$$

with the initial condition $F_0(x) = 1$, $F_1(x) = x + 1$.

(iii) *The minimal polynomial of B has roots all real and distinct.*

Let θ be an eigenvalue of A, and $m(\theta)$ be the multiplicity of θ in A. Then

PROPOSITION 2.3 [7, Appendix; 1]. *If $\theta = k$, then $m(k)=1$. If $\theta \neq k$, i.e. $F_d(\theta)=0$, then*

$$m(\theta)=\frac{nkb_1b_2\cdots b_{d-1}c_2c_3\cdots c_{d-1}}{(k-\theta)F_{d-1}(\theta)F_d'(\theta)},$$

where $F_{d-1}(x)$ is the polynomial defined in Proposition 2 and $F_d'(x)$ is the derivative of $F_d(x)$.

Let $\partial(u,v)$ denote the distance between u and v, and let Γ be *antipodal*, i.e., $\partial(v,w)=d$ for all distinct $v, w \in \Lambda_d(u)$. We construct the *derived graph* Γ' by taking the vertices of Γ' to be the blocks $\{u\} \cup \Lambda_d(u)$ in Γ, two blocks being joined in Γ' whenever they contain adjacent vertices of Γ. Let m be the block size $|\Lambda_d(u)|+1$. The following fact is well known:

PROPOSITION 2.4.

(i) *Γ' also becomes distance-regular. Its valency is k, and its diameter is the integer part of $d/2$.*

(ii) *The intersection matrix of Γ' is the same as the left top quarter of B with the $(d/2-1, d/2)$ entry altered to be $c_{d/2}+b_{d/2}$ if d is even, the $((d-1)/2,(d-1)/2)$ entry altered to be $a_{(d-1)/2}+b_{(d-1)/2}$ if d is odd.*

(iii) *Any two adjacent blocks in Γ' contain m edges of Γ if $d>2$.*

Put labels $1,2,\ldots,m$ on the vertices of each block. Let $d>2$. Then the m edges between two adjacent blocks induce a permutation on $\{1,2,\ldots,m\}$. Therefore the adjacency in Γ can be completely described by attaching a permutation of m letters to each edge of Γ' and giving an orientation to each edge of Γ'. The graph Γ described in this manner is called a *covering graph* of Γ' and denoted by $m\cdot\Gamma'$ [3, Chapter 19].

Let g, g' be the girth of Γ, Γ', respectively. Take an arbitrary circuit of length g' in Γ': $u_0'u_1'u_2'\cdots u_{g'}'$ with $u_0'=u_{g'}'$. Let z_i be the permutation attached to the edge $u_{i-1}'u_i'$. Then it is easy to see the following fact:

PROPOSITION 2.5. *If $g'<g$, then* $z_1^{\epsilon_1}z_2^{\epsilon_2}\cdots z_{g'}^{\epsilon_{g'}}$ fixes no letters, where ϵ_i is 1 or -1 according as the orientation is from u_{i-1}' to u_i' or not.

3. PROOF OF THE THEOREM

Let Γ be a bipartite distance-regular graph with valency 3. By Proposition 2.1, B has the form

$$\left[\begin{array}{cccccccccc}
 & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{r}} & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{s}} & \\
0 & 1 &&&&&&&& \bigcirc \\
3 & 0 & 1 &&&&&&& \\
 & 2 & 0 & \ddots &&&&&& \\
 && 2 & \ddots & 1 &&&&& \\
 &&& \ddots & 0 & 2 &&&& \\
 &&&& 2 & 0 & 2 &&& \\
 &&&&& 1 & 0 & \ddots && \\
 &&&&&& 1 & \ddots & 2 & \\
 &&&&&&& \ddots & 0 & 3 \\
\bigcirc &&&&&&&& 1 & 0
\end{array}\right] \tag{1}$$

with r ones and s twos in the upper diagonal. Clearly it holds that

$$d = r + s + 1. \tag{2}$$

Let $k_i = |\Lambda_i(u)|$ for $i = 0, 1, \dots, d$. Then by Proposition 20.4 of [3], we get

$$\begin{aligned}
k_i &= 3 \times 2^{i-1} \qquad \text{for} \quad i = 1, 2, \dots, r, \\
k_{r+i} &= 3 \times 2^{r-i} \qquad \text{for} \quad i = 1, 2, \dots, s, \\
k_d &= 2^{r-s},
\end{aligned} \tag{3}$$

and

$$n = \sum_{i=0}^{d} k_i = 2(3 \times 2^r - 2^{r-s} - 1).$$

Since k_d is an integer, it holds that

$$r \geqslant s. \tag{4}$$

In what follows, we assume that

$$s \geqslant 1, \tag{5}$$

since the case $s = 0$ has been finished in [5] and [9]. (The uniqueness was personally communicated by N. L. Biggs.)

LEMMA 3.1. *Let $(x-3)F_d(x)$ be the minimal polynomial of B. Then*

(i) *with* $\lambda + \mu = x$, $\lambda\mu = 2$,

$$F_d(x) = \frac{(\lambda+1)(\mu+1)}{(\lambda-\mu)^2}\left[\lambda^{r+s+2} + \mu^{r+s+2} - (\lambda^r + \mu^r)(\lambda^s + \mu^s)\right],$$

and

(ii) *with* $\lambda = \sqrt{2}\, e^{i\alpha}$, $\mu = \sqrt{2}\, e^{-i\alpha}$ $(\alpha \in \mathbb{C})$,

$$F_d(x) = \frac{-2^{(r+s)/2}(x+3)}{2\sin^2\alpha}\left[\cos(r+s+2)\alpha - \cos r\alpha \cos s\alpha\right].$$

Proof. By Proposition 2.2(ii), we get

$$\begin{aligned} F_i(x) &= xF_{i-1}(x) - 2F_{i-2}(x) \qquad \text{for} \quad i = 2,3,\ldots,r, \\ F_{r+1}(x) &= (x+1)F_r(x) - 2F_{r-2}(x), \\ F_{r+i}(x) &= xF_{r+i-1}(x) - 2F_{r+i-2}(x) \qquad \text{for} \quad i = 2,3,\ldots,s, \\ F_{r+s+1}(x) &= (x+1)F_{r+s}(x) - 2F_{r+s-1}(x), \end{aligned} \tag{6}$$

with $F_0(x) = 1$, $F_1(x) = x+1$. If we set

$$T = \begin{pmatrix} 0 & -2 \\ 1 & x \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7}$$

then the recursion (6) can be rewritten as

$$(F_{d-1}(x), F_d(x)) = (1, x+1)T^{r-1}(T+U)T^{s-1}(T+U). \tag{8}$$

The eigenvectors of T are $(1, \lambda)$ and $(1, \mu)$ with eigenvalues λ, μ respectively, where $\lambda + \mu = x$, $\lambda\mu = 2$. Let $E_i(x)$ be the polynomial of degree $i-1$ defined by

$$(E_i(x), E_{i+1}(x)) = (0,1)T^i \qquad \text{for} \quad i = 0,1,2,\dots. \tag{9}$$

Then since

$$(0,1) = \frac{1}{\lambda - \mu}[(1,\lambda) - (1-\mu)],$$

it holds that

$$E_i(x) = \frac{\lambda^i - \mu^i}{\lambda - \mu} \qquad \text{for} \quad i = 0,1,2,\dots. \tag{10}$$

Since $(1, x+1) = (0,1) + (0,1)T$, the equation (8) can be solved in terms of $E_i(x)$ as follows:

$$\begin{aligned}
&(F_{d-1}, F_d)\\
&= [(0,1)+(0,1)T]T^{r-1}(T+U)T^{s-1}(T+U)\\
&= [(E_{r-1}, E_r)+(E_r, E_{r+1})](T+U)T^{s-1}(T+U)\\
&= [(E_r, E_{r+1})+(E_{r+1}, E_{r+2})+(E_r + E_{r+1})(0,1)]T^{s-1}(T+U)\\
&= [(E_{r+s-1}, E_{r+s})+(E_{r+s}, E_{r+s+1})+(E_r + E_{r+1})(E_{s-1}, E_s)](T+U)\\
&= [(E_{r+s}, E_{r+s+1})+(E_{r+s+1}, E_{r+s+2})+(E_r + E_{r+1})(E_s, E_{s+1})]\\
&\quad + [E_{r+s} + E_{r+s+1} + (E_r + E_{r+1})E_s](0,1).
\end{aligned}$$

Therefore we get that

$$F_{d-1}(x) = E_{r+s}(x) + E_{r+s+1}(x) + [E_r(x) + E_{r+1}(x)]E_s(x) \tag{11}$$

and

$$\begin{aligned}
F_d(x) = {}& E_{r+s}(x) + 2E_{r+s+1}(x) + E_{r+s+2}(x)\\
&+ [E_r(x) + E_{r+1}(x)][E_s(x) + E_{s+1}(x)].
\end{aligned}$$

By (10), the equation (11) for $F_{d-1}(x)$ becomes

$$F_{d-1}(x)=\frac{1}{(\lambda-\mu)^2}\left[\lambda^{r+s-1}(\lambda^2-1)(\lambda+2)+\mu^{r+s-1}(\mu^2-1)(\mu+2)\right.$$
$$\left.-\lambda^r\mu^s(\lambda+1)-\lambda^s\mu^r(\mu+1)\right].$$

Since $\lambda+2=\lambda+\lambda\mu=\lambda(\mu+1)$ and $\mu+2=\mu(\lambda+1)$, we get that

$$F_{d-1}(x)=\frac{1}{(\lambda-\mu)^2}\left[\lambda^{r+s}(\lambda^2-1)(\mu+1)+\mu^{r+s}(\mu^2-1)(\lambda+1)\right.$$
$$\left.-\lambda^r\mu^s(\lambda+1)-\lambda^s\mu^r(\mu+1)\right]. \tag{12}$$

If we regard F_{d-1} as a function in x, r, s, i.e., $F_{d-1}(x)=G(x,r,s)$, then we know that $F_d(x)=G(x,r,s)+G(x,r,s+1)$ by (11), and hence by (12) we get that

$$F_d(x)=\frac{1}{(\lambda-\mu)^2}\left[\lambda^{r+s}(\lambda^2-1)(\lambda+1)(\mu+1)+\mu^{r+s}(\mu^2-1)(\lambda+1)(\mu+1)\right.$$
$$\left.-\lambda^r\mu^s(\lambda+1)(\mu+1)-\lambda^s\mu^r(\lambda+1)(\mu+1)\right],$$

which is the desired result (i). The equation (ii) immediately follows by setting $\lambda=\sqrt{2}\,e^{i\alpha}$, $\mu=\sqrt{2}\,e^{-i\alpha}$ $(\alpha\in\mathbb{C})$. ■

As is shown in Lemma 3.1, the minimal polynomial of B is simple enough to find the location of its roots. For $\alpha=i\pi/(d+1)$, $i=1,2,\ldots,d$, with $d=r+s+1$, $F_d(x)$ takes sign $(-1)^{i+1}$ with $x=2\sqrt{2}\cos\alpha$ because the equation (ii) of Lemma 3.1 becomes

$$F_d(x)=c\left[(-1)^i-\cos\frac{ir\pi}{d+1}\cos\frac{is\pi}{d+1}\right]$$

for some negative c, and

$$(-1)^i\cos\frac{ir\pi}{d+1}\cos\frac{is\pi}{d+1}=(-1)^i\cos\frac{i(d-s-1)\pi}{d+1}\cos\frac{is\pi}{d+1}$$
$$=\cos\frac{i(s+2)\pi}{d+1}\cos\frac{is\pi}{d+1}<1.$$

Therefore by the intermediate-value theorem, there exists a root θ_i of $F_d(x)$

such that

$$\theta_i = 2\sqrt{2}\cos\alpha_i \qquad \text{with} \quad \frac{i\pi}{d+1} < \alpha_i < \frac{(i+1)\pi}{d+1}$$

for $i = 1, 2, \ldots, d-1$. Since $F_d(x)$ has degree d, those θ_i together with -3 are all the roots of $F_d(x)$.

LEMMA 3.2. *Let $\theta_0 = 3 > \theta_1 > \theta_2 > \cdots > \theta_{d-1} > \theta_d = -3$ be the eigenvalues of B. Then*

(i) $\theta_i = 2\sqrt{2}\cos\alpha_i$ *with* $\frac{i\pi}{d+1} < \alpha_i < \frac{(i+1)\pi}{d+1}$ *for* $i = 1, 2, \ldots, d-1$,
(ii) $\frac{\pi}{d} < \alpha_1 < \frac{\pi}{r+1}$, *and*
(iii) $\theta_i = -\theta_{d-i}$ *for* $i = 0, 1, 2, \ldots, d$.

Proof. The assertion (i) has just been proved. For the second assertion, we use the trigonometric equality

$$-\cos(r+s+2)\alpha + \cos r\alpha\cos s\alpha = \sin\alpha\sin(r+s+1)\alpha + \sin(r+1)\alpha\sin(s+1)\alpha. \tag{13}$$

Apply the intermediate-value theorem to the right hand side of (13). The last assertion (iii) always holds for bipartite graphs [3, Proposition 8.2], or we can directly verify (iii) by Lemma 3.1. ∎

LEMMA 3.3. *Let θ be an eigenvalue of A with $\theta \neq \pm 3$. The multiplicity of θ in A is given by the formula*

$$m(\theta) = 12n\frac{\sin^2\alpha}{1+8\sin^2\alpha}\,\frac{1}{r+1+(2s\sin^2\alpha+\sin^2 s\alpha)/S(\alpha)}$$

where $\theta = 2\sqrt{2}\cos\alpha$ and

$$S(\alpha) = \frac{1}{4}\left|\varphi^s + 1 - \frac{2}{\varphi}\right|^2 = 2\sin^2(s+1)\alpha - \sin^2 s\alpha + 2\sin^2\alpha$$

with $\varphi = \lambda/\mu = e^{2i\alpha}$.

Proof. We first notice that θ is also an eigenvalue of B and hence can be written as $\theta = 2\sqrt{2}\cos\alpha$ with $0 < \alpha < \pi$. Set

$$\varphi = \frac{\lambda}{\mu} = \frac{\lambda^2}{2}, \quad \chi = \frac{\mu}{\lambda} = \frac{\mu^2}{2} \qquad \text{with} \quad \lambda = \sqrt{2}\,e^{i\alpha}, \quad \mu = \sqrt{2}\,e^{-i\alpha}, \tag{14}$$

i.e.,

$$\varphi + \chi = \frac{\theta^2 - 4}{2}, \qquad \varphi\chi = 1.$$

We shall calculate $m(\theta)$ by using Proposition 2.3.

The equation (12) is rewritten as follows:

$$F_{d-1}(\theta) = \frac{1}{(\lambda-\mu)^2}\{(\lambda+1)[\lambda^{r+s}(\lambda^2-1) + \mu^{r+s}(\mu^2-1) - \lambda^r\mu^s - \lambda^s\mu^r]$$
$$-(\lambda-\mu)[\lambda^{r+s}(\lambda^2-1) - \lambda^s\mu^r]\}.$$

$\lambda^{r+s}(\lambda^2-1) + \mu^{r+s}(\mu^2-1) - \lambda^r\mu^s - \lambda^s\mu^r = 0$ by Lemma 3.1(i), and so

$$F_{d-1}(\theta) = \frac{1}{\lambda-\mu}[-\lambda^{r+s}(\lambda^2-1) + \lambda^s\mu^r],$$

i.e.,

$$F_{d-1}(\theta) = \frac{\lambda^{r+s}}{\lambda-\mu}[\chi^r + 1 - 2\varphi]. \tag{15}$$

By (14), the derivatives of φ and χ with respect to x at $x = \theta$ satisfy

$$\varphi' + \chi' = \theta, \qquad \varphi'\chi + \varphi\chi' = 0.$$

Therefore

$$\varphi' = \frac{\varphi\theta}{\varphi-\chi} \quad \text{and} \quad \chi' = \frac{-\chi\theta}{\varphi-\chi}.$$

Since $\theta/(\varphi-\chi) = 2(\lambda+\mu)/(\lambda^2-\mu^2) = 2/(\lambda-\mu)$, it holds that

$$\varphi' = \frac{2\varphi}{\lambda-\mu} \quad \text{and} \quad \chi' = \frac{-2\chi}{\lambda-\mu}. \tag{16}$$

We rewrite the equation of Lemma 3.1(i) as follows:

$$F_d(x) = \frac{(\lambda+1)(\mu+1)\mu^{r+s}}{(\lambda-\mu)^2}\left[2\varphi^{r+s+1} - \varphi^{r+s} - \varphi^r - \varphi^s - 1 + 2\chi\right]. \quad (17)$$

By (16)

$$F_d'(\theta) = \frac{(\lambda+1)(\mu+1)\mu^{r+s}}{(\lambda-\mu)^2}\,\frac{2}{\lambda-\mu}\big[2(r+s+1)\varphi^{r+s+1} - (r+s)\varphi^{r+s} - r\varphi^r - s\varphi^s - 2\chi\big]$$

$$= \frac{(\lambda+1)(\mu+1)\mu^{r+s}}{(\lambda-\mu)^2}\,\frac{2}{\lambda-\mu}\big[(r+s)\{2\varphi^{r+s+1} - \varphi^{r+s} - \varphi^r - \varphi^s - 1 + 2\chi\} + \{2\varphi^{r+s+1} + s\varphi^r + r\varphi^s + r + s - 2(r+s+1)\chi\}\big].$$

$2\varphi^{r+s+1} - \varphi^{r+s} - \varphi^r - \varphi^s - 1 + 2\chi = 0$, since θ is a root of (17), and hence it holds that

$$F_d'(\theta) = \frac{2(\lambda+1)(\mu+1)\mu^{r+s}}{(\lambda-\mu)^3}\left[2\varphi^{r+s+1} + s\varphi^r + r\varphi^s + r + s - 2(r+s+1)\chi\right],$$

i.e.,

$$F_d'(\theta) = \frac{2(\lambda+1)(\mu+1)\mu^{r+s}}{(\lambda-\mu)^3} \cdot \left[r(\varphi^s + 1 - 2\chi) + s(\varphi^r + 1 - 2\chi) + 2(\varphi^{r+s+1} - \chi)\right]. \quad (18)$$

The product of (15) and (18) is

$$F_{d-1}(\theta)F_d'(\theta) = \frac{2^{r+s+1}(\lambda+1)(\mu+1)}{(\lambda-\mu)^4}(\chi^r + 1 - 2\varphi)(\varphi^s + 1 - 2\chi) \times\left[r + \frac{s(\varphi^r + 1 - 2\chi)}{\varphi^s + 1 - 2\chi} + \frac{2(\varphi^{r+s+1} - \chi)}{\varphi^s + 1 - 2\chi}\right]. \quad (19)$$

SUBLEMMA. *It holds that*

$$(\chi^r+1-2\varphi)(\varphi^s+1-2\chi)=-2(\varphi+\chi-2)=8\sin^2\alpha, \tag{20}$$

$$\frac{\varphi^r+1-2\chi}{\varphi^s+1-2\chi}=\frac{-2(\varphi+\chi-2)}{|\varphi^s+1-2\chi|^2}=\frac{8\sin^2\alpha}{|\varphi^s+1-2\chi|^2}, \tag{21}$$

$$\frac{\varphi^{r+s+1}-\chi}{\varphi^s+1-2\chi}=-\frac{(\varphi^{s+1}+\chi^{s+1}-2)+(\varphi+\chi-2)}{|\varphi^s+1-2\chi|^2}$$

$$=\frac{4\sin^2(s+1)\alpha+4\sin^2\alpha}{|\varphi^s+1-2\chi|^2}, \tag{22}$$

and

$$\tfrac{1}{4}|\varphi^s+1-2\chi|^2=2\sin^2(s+1)\alpha-\sin^2 s\alpha+2\sin^2\alpha. \tag{23}$$

Proof. We put θ in x of (17) and get

$$2\varphi^{r+s+1}-\varphi^{r+s}-\varphi^r-\varphi^s-1+2\chi=0.$$

Solve this identity for φ^r. Then we get

$$\varphi^r=-\frac{1}{\varphi^s}\frac{\varphi^s+1-2\chi}{\chi^s+1-2\varphi}. \tag{24}$$

Put (24) in $\varphi^r+1-2\chi$. Then we get

$$\varphi^r+1-2\chi=\frac{-2(\varphi-2+\chi)}{\chi^s+1-2\varphi}. \tag{25}$$

If we take complex conjugates, φ and χ are interchanged and so the identity (25) becomes (20). The identity (21) immediately follows from (25) divided by $\varphi^s+1-2\chi$. By (24), we get

$$\varphi^{r+s+1}=-\varphi\frac{\varphi^s+1-2\chi}{\chi^s+1-2\varphi},$$

and so

$$\frac{\varphi^{r+s+1}-\chi}{\varphi^s+1-2\chi}=-\frac{\varphi}{\chi^s+1-2\varphi}-\frac{\chi}{\varphi^s+1-2\chi}.$$

The identity (22) follows from this. For each $i\in\mathbb{Z}$,

$$2-(\varphi^i+\chi^i)=2-2\cos 2i\alpha=4\sin^2 i\alpha. \tag{26}$$

Therefore each of (20), (21), (22), (23) is expressed in terms of trigonometric functions. For example,

$$\begin{aligned}|\varphi^s+1-2\chi|^2&=(\varphi^s+1-2\chi)(\chi^s+1-2\varphi)\\&=2(2-\varphi^{s+1}-\chi^{s+1})-(2-\varphi^s-\chi^s)+2(2-\varphi-\chi)\\&=8\sin^2(s+1)\alpha-4\sin^2 s\alpha+8\sin^2\alpha,\end{aligned}$$

and we get (23). This completes the proof of the Sublemma. ■

Since $(\lambda+1)(\mu+1)=\theta+3$ and $(\lambda-\mu)^2=\theta^2-8=-8\sin^2\alpha$, the identity (19) becomes

$$\begin{aligned}F_{d-1}(\theta)F_d'(\theta)&=\frac{2^{r+s+1}(\theta+3)}{8\sin^2\alpha}\left[r+\frac{2s\sin^2\alpha}{S(\alpha)}+\frac{2\sin^2(s+1)\alpha+2\sin^2\alpha}{S(\alpha)}\right]\\&=\frac{2^{r+s+1}(\theta+3)}{8\sin^2\alpha}\left[r+1+\frac{2s\sin^2\alpha+\sin^2 s\alpha}{S(\alpha)}\right]\end{aligned} \tag{27}$$

with $S(\alpha)=\frac{1}{4}|\varphi^s+1-2\chi|^2$. By Proposition 2.3,

$$m(\theta)=\frac{3n\times 2^{r+s}}{(3-\theta)F_{d-1}(\theta)F_d'(\theta)}. \tag{28}$$

This together with (27) proves Lemma 3.3. ■

LEMMA 3.4. *Let $\theta_0=3>\theta_1>\theta_2>\cdots>\theta_{d-1}>\theta_d=-3$ be the eigenvalues of A as in Lemma 3.2. If $r\geqslant 8$, then there exists some θ_i $(2\leqslant i\leqslant d-2)$ such that*

(i) $8-\theta_i^2=8\sin^2\alpha_i>1$, *and*
(ii) $m(\theta_1)=m(\theta_i)$.

Proof. Since $F_d(x)$ is monic with integer coefficients, all the θ_i are algebraic integers. Therefore the product

$$\Pi\left(8-\theta_i^2\right) \tag{29}$$

over all the θ_i algebraic conjugate to θ_1 is an integer. Since $8-\theta_i^2 = 8\sin^2\alpha_i > 0$ by Lemma 3.2, the product (29) is positive and hence greater than or equal to 1, whereas by Lemma 3.2(ii)

$$8-\theta_1^2 = 8\sin^2\alpha_1 < 8\sin^2\frac{\pi}{r+1} < 1 \qquad \text{if} \quad r \geqslant 8.$$

This implies that there exists some θ_i algebraic conjugate to θ_1 such that $8-\theta_i^2 = 8\sin^2\alpha_i > 1$. Since θ_i is algebraic conjugate to θ_1, their multiplicities in A are the same, i.e., $m(\theta_1) = m(\theta_i)$. ■

LEMMA 3.5. *Let* $\theta = 2\sqrt{2}\cos\alpha$ $(0<\alpha<\pi)$ *be an eigenvalue of* A. *Then*

(i) $m(\theta) < \dfrac{12n}{r+1}\,\dfrac{\sin^2\alpha}{1+8\sin^2\alpha}$, *and*

(ii) $m(\theta) > \dfrac{3n}{4}\,\dfrac{1}{r+1+(s+4)(3+2\sqrt{2})}$ *if* $8\sin^2\alpha > 1$.

Proof. The first assertion is trivial by Lemma 3.3. To bound $m(\theta)$ from below, we estimate $S(\alpha)$. First we observe that

$$\left|1-\frac{2}{\varphi}\right| = \left|1-2\cos 2\alpha + 2\sqrt{-1}\,\sin 2\alpha\right| = \sqrt{1+8\sin^2\alpha}\,.$$

Therefore

$$\begin{aligned} S(\alpha) &= \frac{1}{4}\left|\varphi^s + 1 - \frac{2}{\varphi}\right|^2 \\ &\geqslant \frac{1}{4}\left(\sqrt{1+8\sin^2\alpha} - 1\right)^2 \end{aligned}$$

and so

$$\frac{1}{S(\alpha)} \leqslant \frac{4\left(\sqrt{1+8\sin^2\alpha}+1\right)^2}{\left(8\sin^2\alpha\right)^2} \tag{30}$$

Replace $1/S(\alpha)$ in the formula of Lemma 3.3 by the inequality (30). Then we get

$$m(\theta) \geqslant 12n \frac{\sin^2\alpha}{1+8\sin^2\alpha} \frac{1}{r+1+\dfrac{4(2s\sin^2\alpha+1)\left(\sqrt{1+8\sin^2\alpha}+1\right)^2}{(8\sin^2\alpha)^2}}.$$

The right hand side of the above inequality increases with $\sin^2\alpha$, and hence if $8\sin^2\alpha > 1$, we get

$$m(\theta) > 12n \frac{\frac{1}{8}}{1+1} \frac{1}{r+1+(s+4)\left(\sqrt{2}+1\right)^2},$$

which is the desired result. ■

LEMMA 3.6. *If* $r \geqslant 8$, *then*

$$s+4 > \frac{r+1}{2\left(3+2\sqrt{2}\right)} \left(\frac{1}{8\sin^2\pi/(r+1)} - 1 \right).$$

Proof. Take θ_i as in Lemma 3.4. Then by Lemma 3.5,

$$m(\theta_1) < \frac{12n}{r+1} \frac{1}{8+\dfrac{1}{\sin^2\pi/(r+1)}}$$

and

$$m(\theta_i) > \frac{12n}{16} \frac{1}{r+1+(s+4)\left(3+2\sqrt{2}\right)}.$$

[We have used the inequality $\alpha_1 < \pi/(r+1)$ in Lemma 3.2 as well.] Therefore

$$1 = \frac{m(\theta_i)}{m(\theta_1)} > \frac{1}{16}\left(8+\frac{1}{\sin^2\pi/(r+1)}\right) \frac{1}{1+\dfrac{s+4}{r+1}\left(3+2\sqrt{2}\right)}.$$

Solve this inequality for $s+4$. Then we get the desired result. ■

By (4), $r \geqslant s$, and the previous lemma,

$$r+4 > \frac{r+1}{2(3+2\sqrt{2})}\left(\frac{1}{8\sin^2 \pi/(r+1)} - 1\right),$$

i.e.

$$1 > \frac{1}{2(3+2\sqrt{2})}\left(\frac{1}{8\sin^2 \pi/(r+1)} - 1\right) - \frac{3}{r+1}. \tag{31}$$

The right hand side of (31) increases with $r+1$ and becomes greater than 1 when $r = 32$. Therefore we get

$$r \leqslant 31. \tag{32}$$

The admissible (r, s) for Lemma 3.6 are as follows:

r	31	30	29	28	27	26	25	
$r \geqslant s \geqslant$	29	26	23	21	18	16	13	
r	24	23	22	21	20	19	18	17
$r \geqslant s \geqslant$	11	10	8	6	5	4	2	1

(33)

and all (r, s) with $16 \geqslant r \geqslant 1$ and $r \geqslant s \geqslant 1$.

In order to eliminate the remaining finite cases listed in (33), we count the number of circuits of length $2(r+1)$, $2(r+2)$, and $2(r+3)$. Let

$$c_q \text{ be the number of circuits of length } q. \tag{34}$$

Lemma 3.7. *It holds that*

(i) $c_{2(r+1)} = 3 \times 2^{r-1} \times n/2(r+1)$,

(ii) $c_{2(r+2)} = \begin{cases} 3 \times 2^r \times n/2(r+2) & \text{if } s \geqslant 2 \\ 3 \times 2^{r+1} \times n/2(r+2) & \text{if } s = 1 \text{ and } r \geqslant 2 \end{cases}$, *and*

(iii) $c_{2(r+3)} = \begin{cases} 3 \times 7 \times 2^{r-1} \times n/2(r+3) & \text{if } s \geqslant 3 \text{ and } r \geqslant 4, \\ 3 \times 11 \times 2^{r-1} n/2(r+3) & \text{if } s = 2 \text{ and } r \geqslant 4, \\ 3 \times 5 \times 2^{r-1} n/2(r+3) & \text{if } s = 1 \text{ and } r \geqslant 4. \end{cases}$

Proof. Count in two ways the number of pairs (u, C), where C is a circuit of length $2(r+1)$ containing u. Then we get the formula (i).

Count in two ways the number of pairs (u, C_1) and triples (u, v, C_2), where C_1 is a circuit of length $2(r+2)$ containing u, $\{u, v\}$ is an edge, and C_2

is a circuit of length $2(r+1)$ containing v but not u. Then we get the formula (ii).

Count in two ways the number of pairs (u, C_1), triples (u, v, C_2), and quadruples (u, v, w, C_3), where C_1 is a circuit of length $2(r+3)$ containing u, $\{u, v\}$ and $\{v, w\}$ are edges, C_2 is a circuit of length $2(r+2)$ containing v but not u, and C_3 is a circuit of length $2(r+1)$ containing w but not u or v. Then we get the formula (iii). ■

The (r, s) in (33) which satisfy the integer condition of Lemma 3.7 are

r	30	23	15	12	7	6	5	4	2	1
s	29	12	8	11	4	5,2	3	3	2,1	1

(35)

For (r, s) in (35), we check the feasibility condition, i.e. $m(\theta) \in \mathbb{Z}$, and get the θ which violate the feasibility. They are listed in Table 2, where $\omega = \varphi + \chi = (\theta^2 - 4)/2$.

All the equations for ω are irreducible in the list. We take a primitive root of the cyclotomic equations for φ. Those ω, φ determine θ^2, and we can see from the formula for $m(\theta)$ that the value of θ^2 is enough to determine $m(\theta)$ (or by Proposition 8.2 of [3], it holds that $m(\theta) = m(-\theta)$ for every bipartite graph). Those $m(\theta)$ are not integers [in fact they are not even rational numbers, except the case $(r, s) = (5,3), (6,2)$], which is a contradiction.

Thus the possible parameters are only

r	4	2	2	1
s	3	2	1	1

(36)

For each of the above parameters (r, s), we shall give a brief proof of the

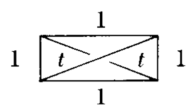
TABLE 2

(r, s)	θ
(5,3)	$\varphi^3 - 1 = 0$
(6,2)	$\varphi^3 - 1 = 0$
(6,5)	$2\omega^3 + \omega^2 - 5\omega - 1 = 0$
(7,4)	$2\omega^4 + \omega^3 - 7\omega^2 - 2\omega + 5 = 0$
(12,11)	$\varphi^{12} - 1 = 0$
(15,8)	$\varphi^8 - 1 = 0$
(23,12)	$\varphi^{12} - 1 = 0$
(30,29)	$\varphi^{30} - 1 = 0$

existence and uniqueness of Γ. We first observe that Γ is antipodal. This is because $\partial(v,w) \geqslant 2(s+1)$ for all distinct $v, w \in \Lambda_d(u)$ and $2(s+1) = r+s+1 = d$ for (r,s) in the list (36) (if $r=s$, then $|\Lambda_d(u)|=1$ and there is nothing to prove). By Proposition 2.4, the intersection matrix of the derived graph Γ' is

$$B' = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \\ 3 & 0 & 3 \\ & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \\ 3 & 0 & 1 \\ & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & & \\ 3 & 0 & 1 & & \\ & 2 & 0 & 1 & \\ & & 2 & 0 & 3 \\ & & & 2 & 0 \end{pmatrix}, \tag{37}$$

TABLE 3



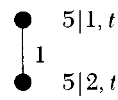
$t = (1,2)$

(ii) $2 \cdot K_4 \cong$ cube

1	1	1
1	Z	Z^{-1}
1	Z^{-1}	Z

$z = (1,2,3)$

(iv) $3 \cdot K_{3,3}$



$t = (1,2)$

(vi) $2.O_3$

1	1	1												
1			1	1										
1					1	1								
	1						1	1						
	1								1	1				
		1									1	1		
		1											1	1
			1				Z				Z^{-1}			
			1							Z^{-1}				Z
				1					Z				Z^{-1}	
				1				Z^{-1}				Z		
					1		Z^{-1}						Z	
					1					Z		Z^{-1}		
						1			Z^{-1}		Z			
						1		Z						Z^{-1}

(viii) $3 \cdot$(Tutte's 8-cage) $z = (1,2,3)$

for $(r, s) = (1,1)$, $(2,1)$, $(2,2)$, $(4,3)$, respectively. The distance-regular graph Γ' is uniquely determined to be $K_4, K_{3,3}, O_3$ (Petersen's graph) and the generalized 4-gon (Tutte's 8-cage), respectively. In each case, it holds that $g' < g = 2(r+1)$. Therefore $z_1^{\epsilon_1} z_2^{\epsilon_2} \cdots z_{g'}^{\epsilon_{g'}}$ in Proposition 2.5 is fixed-point-free for every circuit of length g'. This determines Γ uniquely as shown in Table 3 (cf. [8]). In the diagrams in the table, the orientations are from one half to the other if Γ' is bipartite and arbitrary if $m = 2$. The diagram for (vi) indicates that the identity permutation is attached to each "spoke" and the transposition $(1,2)$ is attached to everywhere else (cf. [3, p. 152]).

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